Conditional calculus on fractal structures and

its application to galaxy distribution

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Abstract It is shown that calculus can apply on a fractal structure with the condition that the infinitesimal limit of change of the variable is larger than the lower cut-off of the fractal structure, and an assumption called local decomposability. As an application, it is shown that the angular projection of a fractal distribution in 3-dimensional space is not homogeneous at sufficiently large angles. Therefore the angular projection of galaxy distribution for sufficiently large angles can discriminate the fractal and the homogeneity pictures.

It seems that calculus cannot apply in a fractal structure, which possesses infinite places of discontinuity. However, it is shown in this letter that it can be done in a conditional way. Just as that only with calculus can we deal with manifold and function with an arbitrary shape, the conditional calculus on fractals will enable us deepen the studies of fractal.

This work is partly motivated by the investigations on fractal distribution of galaxies. While it is an agreement that the galaxy distribution approximates a fractal over a considerable range of scales, it is under a debate whether it becomes homogeneous on a scale about, say, $20h^{-1}Mpc$ [1] [2], or is fractal up to the present observational limit [3][4]. A crucial problem, which is also controversial, related to the above question is whether the angular projection of a fractal embedded in a 3-dimensional (3d) space is homogeneous. Since before the extensive redshift survey the galaxy catalogs were for angular coordinates and now there is much angular information [2], a difference in angular projection will help us choose between the two alternative pictures. Based on a numerical simulation on a fractal structure geneated by a Levy flight in 3d, it was stated that the angular projection shows power-law correlation at small angles but becomes homogeneous at large angles [4]. However, as we pointed out in a previous work [5], their

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explaination is inconsitent, taking the implicit assumption of angular homogeneity in distribution. We distinguished the concepts of angular distribution and angular projection, the former refers to the number of points as a function of polar angle within a given radial depth while the latter is defined through the solid-angular density represented as a function of polar angle. It was shown that the power-laws for angular projection at small angles are manifestinations of the so-called local angular fractal, which refers to that for a small conic part of a sphere defined by the angle 2θ and the depth L, when θ is small enough, the points on each spherical shell within the conic part have fractal distribution. The local angular fractal may reconcil the evidence of fractal and that claimed to be of homogeneity, and realize Cosmological Principle in a fractal structure in a rather strict way. The behavior at large angles remained open there, and will be obtained here through the application of conditional calculus. Methods of calculus is also expected when one deals with the fractal structure on a curved space, which should be taken into account for large-scale structure of the universe.

To define a function on a fractal, there are two alternative sets of independent variables, one is the continuous variables, or in geometric interpretation, coordinates of the continuous space on which the fractal is embedded, another

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is the variables or coordinates restricted on the fractal, the inner coordinates, effectively they form a continuous subspace. A fractal is only a fractal looked in the embedding space, so we do not know whether the latter viewpoint is useful, practically one takes the former.

Let us start with the derivative of $f(x)$, $f'(x) = \lim_{\Delta x \to 0}[f(x + \Delta x) - f(x)]/\Delta x$. For it to be meaningful, $f(x + \Delta x)$ should be within the fractal set, therefore Δx should conditionally approach 0 so that it is larger than the lower cut-off size of the fractal structure. Therefore we have the following condition of infinitesimal.

Condition. The infinitesimal limit of change of variable should be taken in such a way that it is larger than the lower cut-off size of the fractal structure.

A mathematically-ideal fractal is infinitely iterated and there is no lower cut-off, the above condition is automatically satisfied. Practically, this means that the fractal under study should be large enough. The derivative on a fractal has been used, e.g., in defining the conditional density [4], where the above condition is actually an implicit assumption. So there is almost no difficulty in using derivative.

The difficulty lies in integral. How can we integrate a function $f(x)$ only over the points belonging to the fractal? In the case of 1-dimensional

embedding space, it can be done through change of variable. Suppose $x_f = c_x x^{D_x}$, where the subscript "f" denotes the variable on the fractal, c_x is a coefficient and D_x is the fractal dimension, then $dx_f = c_x D_x x^{D_x - 1} dx$, the integral is thus

$$I = \int_{x_f(a)}^{x_f(b)} f(x) dx_f = c_x D_x \int_a^b f(x) x^{D_x - 1} dx, \tag{1}$$

where a and b are the limits of the integration.

Now we entend it to multiple integral, which can only be done by transformed to an iterated one. In the ordinary case of continuous space this is based on expressing the region of integration in terms of the infinitesimal changes of the independent variables, in geometric interpretation, the element of volume is expressed in the infinitesimal changes of curvilinear coordinates. On a fractal, the element of volume can be expressed as the product of the variables on the fractal, then transformed to those of the embedding continuous space. Here the following assumption is needed.

Assumption. The infinitesimal element of volume on a fractal is a product of infinitesimal changes of lengths in orthogonal directions, these infinitesimal changes are fractal subsets of the infinitesimal element of volume.

This can be called the assumption of local decomposability. It can be

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seen that this assumption is based on the condition of infinitesimal. It is most likely that under this condition the above assumption can be valid, especially in many random fractals such as that generated by a Levy flight and the galaxy distribution on the scales exhibiting fractal structure. The sum of the dimensions of the fratal subsets is, of course, the total dimension of the fractal. But we have no reason that the the decomposition is independent of the position, the dimensions of the subfractal may depend on position. If they are independent of the position, the fractal can be referred to as being uniform.

Thus the multiple integral on a fractal can be done as an iterated integral over the subfractals, with the variables then changed to those of embedding continuous space. For example, the volume integral in terms of the Cartesian coordinates is $\int \int \int_V f(x) dV_f = \int \int \int f(x) dx_f dy_f dz_f = c_x c_y c_z \int \int \int f(x) x^{D_x - 1} y^{D_y - 1} z^{D_z - 1} dx dy dz$. In this way all calculus can apply to fractals.

Formally, calculus may apply with the variables of fractals by transforming derivative with the continuous variable to that with the inner variable of the fractal, $i.e.$, $df(x_f)/dx_f = c_x^{-1} x^{1 - D_x} df[x_f(x)]/dx$. For expample, this derivative of number of points is a constant, showing the fractal subset form

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an effective homogeneous space. Seen from the embedding space, this derivative is fractional.

Now we turn to the angular projection of a conic part defined by L and 2θ, of a fractal in 3-dimensional space. In terms of spherical coordinates (r, θ, ϕ), the number of points in this volume is

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\begin{aligned}
N(L, 2\theta) &= \int_0^L \int_0^\theta \int_0^{2\pi} \partial_r(c_r r^{D_r}) \partial_\theta[c_\theta(r\theta)^{D_\theta}] \partial_\phi[c_\phi(r\sin\theta\phi)^{D_\phi}] \\
&= AL^D \cdot \int_0^\theta \theta^{D_\theta - 1}(\sin\theta)^{D_\phi} d\theta,
\end{aligned}
\tag{2}
$$

where $A = c_r c_\theta c_\phi D_r D_\theta (2\pi)^{D_\phi}$, D_r, D_θ, D_ϕ and D are fractal dimensions of the infinitesimal subfractals and the total fractal, respectively. ∂_r, ∂_θ and ∂_ϕ represent partial derivatives. The second equality is valid when D_r, D_θ and D_ϕ are constants. When θ is small, we obtain $N(L, 2\theta) = AL^D \theta^{D_\theta + D_\phi}$, which is just the "local angular fractal" discussed in [5]. If an arbitrary azimuthal angle ϕ is considered, the number of points are then proportional to ϕ^{D_ϕ}. We suggest this be tested for galaxy distribution.

For an isotropic fractal, all directions are equivalent each other, so the dimension of the subfractal in each direction is $D/3$, i.e. $D_r = D_\theta = D_\phi = D_x = D_y = D_z = D/3$. This is most likely satisfied by the galaxy distribution where $D \approx 2$ while $D_\theta \approx 1.3$ [3] [5].

The solid angle is $2\pi(1-\cos\theta)$, so the conditional angular density defined from the origin is $\Gamma(\theta) = dN(\theta)/d\Omega(\theta) = (dN/d\theta)/(d\Omega/d\theta) = (A/2\pi)L^D F(\theta)$, where $F(\theta) = \theta^{D_\theta-1}(\sin\theta)^{D_\phi-1}$. If the sample size is characterized by $2\theta_M$ and L, the average angular density over the sample is $< n >_M = N(\theta_M)/\Omega(\theta_M)$, which is surely dependent on θ_M. The angular correlation function is $\omega(\theta) = \Gamma(\theta)/ < n >_{\theta_M} -1$, which is dependent on θ through $\Gamma(\theta)$, and also on θ_M. But since it breaks down spuriously at angles much smaller than θ_M [4][5], the dependence of angular projection on θ should be examined in $\Gamma(\theta)$, thus just $F(\theta)$.

The dependence of $F(\theta)$ on θ can be seen from Figure 1, there is a power-law region at small angles, then there is a relatively flat region, which is not so short since the plot is a log-log one, but finally it increases with θ. So the conclusion is that the angular projection of fractal is not homogeneous.

For the fractal generated by a Levy flight [4] is isotropic implied by the generation rule. For the galxy distribution, there is also much evidence of isotropy [2]. So the dimension of infinitesimal subfractals are likely to be indeed independent on the position (cf. discussions in [5]). Even if they are dependent on position, the qualitative nature of dependence of the conditional angular density on the angle is impossible to change. Though we

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obtain the result disagreeing with the claim in [4], it is not contradictory with what was actually reported, only a very small region was given there, and after a fairly flat region of $\sim 0.3°$, there can be observed an indication of increase. So the claimed homogeneity of angular projection at large angles is quite unsure from there.

So the behavior of conditional angular density at sufficiently large angles can discriminate fractal model and homogeneous model for the galaxy distribution on the corresponding scale. Unfortunately, to our knowledge, there is still no such evidence. It should be noted that one should use $\Gamma(\theta)$ instead of $\omega(\theta)$ since $\omega(\theta)$ decreases rapidly at angles much smaller than θ_M. Among the interesting problems also is that to investigate in various fractal structures and the galaxy distribution whether the assumption of local decomposability is indeed valid, and the dependence or independence of the dimensions of local subfractals on the position.

References

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Figure Caption:

Figure 1. Log-log plot of $F(\theta) = \theta^{D_\theta - 1} (\sin \theta)^{D_\phi - 1}$ by setting $D_\theta = 0.6$ and $D_\phi = 0.7$ so that $D_\theta + D_\phi$ equals what was estimated for galaxy distribution [5].